NORTH-HOLLAND

A Joint Estimator for the Eigenvalues of the Reproduction Mean Matrix of a Multitype Galton-Watson Process

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ABSTRACT

A joint estimator for the eigenvalues of the reproduction mean matrix of a nonsingular, positively regular, supercritical, multitype Galton-Watson process is proposed. The estimator is based exclusively upon the vectors of the numbers of elements of each type in the successive generations. Conditions for the almost sure convergence are studied. © 1997 Elsevier Science Inc.

1. INTRODUCTION

Let $\{Z_N = (Z_N(1), \dots, Z_N(r)),\ N = 0, 1, \dots,\}$ be a nonsingular, positively regular Galton-Watson process with r types. Suppose also (see [3, Chapter V] or [6] for that usual notation), that ρ, the principal eigenvalue of the reproduction mean matrix M, is bigger than 1 (supercritical case) and that u and v, the associated right and left eigenvectors, are normalized so that $u \cdot v = v \cdot \mathbf{1} = 1$.

Though some work continues to be done on problems related to the estimation of the Perron root ρ and its left eigenvector v (see, e.g., [8, 9]), the problem of estimating all the eigenvalues of M seems not to have been treated. However, some knowledge about the spectrum of M would be interesting; in fact, it is well known [2] that the asymptotic behavior of some

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estimators of ρ based on the total generation sizes depends, to some extent, on the relative sizes of ρ and the square of the absolute value of some other eigenvalue of M.

In the case where the observation of the process includes records of all the individual offspring sizes, up to the Nth generation, the eigenvalues of the maximum likelihood estimator of M provide immediately a joint estimator, converging almost surely to these parameters, taking into account the continuity of the function mapping the entries of a square real matrix on its eigenvalues (see Appendix D of [7]).

In the present paper we present an answer to the problem when only the successive generation vectors Z_N are observed.

The proposed estimator is seen to converge almost surely in the particular case where, for every nonprincipal eigenvalue ζ of M, we have $|\zeta|^2 > \rho$. For the sake of brevity, the results leading to this proof, which is the substantial part of Section 3, are only presented with real ζ, but the case where some or all the nonprincipal eigenvalues are complex can also be considered, needing only some additional computation.

In Section 2, we establish some auxiliary results concerning the convergence of the martingale $Z_N M^{-N}$, which plays a fundamental role in the proof and recall a few results on matrix theory used in the context. To avoid making trivial exceptions on the set of extinction and ensure most of the convergences we will be dealing with, we are also assuming, throughout, the commonly used hypotheses:

(i) $q_i = P^i[Z_N = \mathbf{0}$ for some $N] = 0$, $i = 1, \ldots, r$ (extinction probability vector $q = \mathbf{0}$).

(ii) all the entries of Σ^i, the covariance matrix of the reproduction vector of type i, are finite, i.e., $\Sigma^i_{jk} = \operatorname{cov}^i(Z_1(j), Z_1(k)) < +\infty$ $(i, j, k = 1, \ldots, r)$.

In Section 4 we give some preliminary simulation examples showing how the convergence occurs and discuss some problems connected with the actual computation of the estimator.

Finally we want to point out that the condition $|\zeta|^2 > \rho$ mentioned above, though restricting the practical use of the estimator, can be understood, in certain cases, in terms of the reproduction pattern of the population using the following corollary of the Geršgorin disc theorem [7].

PROPOSITION 1.1. *Let $A = [a_{ij}]$ be a complex $n \times n$ matrix. Then all the eigenvalues of A lie in the region*

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}$$

This indicates, roughly, that good candidates for the use of this estimation process should be populations where types do not evolve in a radical way, in other words, where there is only mild parent-to-child changing of types.

2. SOME AUXILIARY RESULTS

We begin by establishing a proposition that groups some known results and emphasizes a special behavior of the limit of the linear functionals $Z_N \cdot \tilde{u}$, where $\tilde{u}$ is a right eigenvector of M corresponding to any nonprincipal eigenvalue ζ.

PROPOSITION 2.1. *If for a nonprincipal eigenvalue ζ of M we have $|\zeta|^2 > \rho$, then there exists a random variable $\tilde{W}$ such that*

$$\lim_{N \to +\infty} \frac{Z_N \cdot \tilde{u}}{\zeta^N} = \lim_{N \to +\infty} \tilde{W}_N = \tilde{W} \qquad (a.s., q.m.);$$

moreover, the random variable $\tilde{W}$ is almost surely bounded away from zero, except for special structures of the Σ^i.

Proof. First note that, according to Theorem 4′, of [3, p. 194], $\tilde{W}_N$ is a martingale.

Consider, using the theory of squared integrable martingales (cf. [4]), the increasing process associated with $\tilde{W}_N$ given by

$$A_n = \sum_{n=1}^{N} E\left[\left(\frac{(Z_n - Z_{n-1}M)\cdot \tilde{u}}{\zeta^n}\right)^2\right] \Big/ Z_{n-1}$$

$$= \sum_{n=1}^{N} \frac{Z_{n-1} \cdot b}{(\zeta^n)^2},$$

where $b = \operatorname{var}[Z_1 \cdot \tilde{u}]$ is fixed with components $b_i = \tilde{u}\Sigma^i \tilde{u}^T$ and $Z_{n-1} \cdot b \geqslant 0$ $(n \geqslant 1)$. If $\rho/|\zeta|^2 < 1$ as assumed, the almost sure convergence is proved, since it is easy to show that $P\{A_N < +\infty\} = 1$, using the classical result

$$\frac{Z_n \cdot b}{\rho^n} \xrightarrow[n \to +\infty]{\text{a.s.}} (v \cdot b) W < +\infty \qquad \text{(a.s.)}.$$

Analogously, we find that

$$E[A_\infty] = \lim_{N \to +\infty} \frac{1}{\zeta^2} \sum_{n=1}^{N} E\left[\frac{Z_{n-1} \cdot b}{\rho^{n-1}}\right]\left(\frac{\rho}{\zeta^2}\right)^{n-1},$$

and since we have

$$\lim_{n \to +\infty} E\left[\frac{Z_n \cdot b}{\rho^n}\right] < +\infty,$$

we conclude that $E[A_\infty] < +\infty$, which implies the convergence in quadratic mean.

Let us now treat the second part of the proposition.

Note that if $\zeta > 0$ the result is stated exactly in this form in [1], and again in [2], when the limiting behavior of the functionals $Z_N \cdot a$ with $\lambda^2 > \rho$ is considered. In fact, using their notation, if $a = \tilde{u}$ we have $\tilde{u} \cdot v = 0$, $\lambda = \lambda(\tilde{u}) = \zeta$, $\gamma(\tilde{u}) = 1$, and

$$\frac{Z_N \cdot a}{\lambda^N N^{(\gamma - 1)}} = \tilde{W}_N.$$

So

$$\lim_{N \to +\infty} \frac{Z_N \cdot a}{\lambda^N N^{(\gamma-1)}} - H_N = 0 \qquad \text{(a.s.)}, \tag{1}$$

together with $\liminf |H_N| \neq 0$, enables us to conclude (except for special structures of Σ^i), the nonnullity of $\lim \tilde{W}_N$ whose existence was just proved above.

When $\zeta < 0$, the first sequence in (1) turns out to be

$$\frac{Z_N \cdot \tilde{u}}{|\zeta|^N}. \tag{2}$$

Noting that for the points satisfying

$$\lim_{N \to +\infty} \frac{Z_N \cdot \tilde{u}}{\zeta^N} = 0$$

the limit of (2) exists and is also null, we can also conclude, using the preceding arguments, that $\tilde{W} \neq 0$ (a.s.). ∎

Having established these results, we can now focus on the ones preparing the study of our estimator.

LEMMA 2.1. *The sequence* $\{T_N\}_{N \geqslant 0} = \{Z_N \cdot M^{-N}\}_{N \geqslant 0}$ *is vectorial martingale with respect to* $\{\mathscr{F}_N\}$, $\mathscr{F}_N = \sigma(Z_0, Z_1, \ldots, Z_N)$.

Proof. Clearly we have

$$E[T_N/\mathscr{F}_0] = Z_0 M^N M^{-N} = Z_0$$

and

$$E[T_N/\mathscr{F}_{N-1}] = Z_{N-1} M M^{-N} = Z_{N-1} M^{-(N-1)} = T_{N-1}. \qquad ∎$$

Before proceeding to the next lemma, we recall that the Jordan normal form of M enables us to say that each complex vector x can be uniquely written as

$$x = \sum_{\nu=1}^{\nu_0} \sum_{j=1}^{j_0(\nu)} (x \cdot u_{\nu,j}) v_{\nu,j}, \tag{3}$$

where ρ_ν are the different eigenvalues of M and $u_{\nu,j}, v_{\nu,j}$ $[\nu = 1, \ldots, \nu_0, j = 1, \ldots, j_0(\nu)]$ are possibly complex vectors such that

$$Mu_{\nu,1} = \rho_\nu u_{\nu,1}, \qquad Mu_{\nu,j} = u_{\nu,j-1} + \rho_\nu u_{\nu,j}, \qquad j = 2, \ldots, j_0(\nu),$$

$$v_{\nu,j_0(\nu)} M = \rho_\nu v_{\nu,j_0(\nu)}, \qquad v_{\nu,j} M = v_{\nu,j+1} + \rho_\nu v_{\nu,j}, \qquad j = 1, \ldots, j_0(\nu) - 1$$

LEMMA 2.2. *The almost sure limit of the sequence* T_N, *i.e.*, $T = \lim_{\text{a.s.}} T_N$, *exists if for every nonprincipal eigenvalue* ζ *of* M *we have* $|\zeta|^2 > \rho$.

Proof. Using (3) we can write

$$Z_N M^{-N} = \sum_{\nu=1}^{\nu_0} \sum_{j=1}^{j_0(\nu)} \left(Z_N M^{-N} \cdot u_{\nu,j}\right) v_{\nu,j},$$

and study the almost sure convergence of $Z_N M^{-N}$ using these components. We begin by noting that for a certain $\nu = \mu$, $\rho_\mu = \rho$. In that case, $j_0(\mu) = 1$, $u_\mu = u$, $v_\mu = v$, and we have the well-known classical result

$$\lim_{N \to +\infty} \frac{Z_N \cdot u}{\rho^N} = W \qquad \text{(a.s., q.m.)}$$

For $\nu \neq \mu$ we have, using the convergence part of Proposition 1,

$$\lim_{N \to +\infty} Z_N M^N \cdot u_{\nu,1} = \lim_{N \to +\infty} \frac{Z_N \cdot u_{\nu,1}}{\rho^N} = W_{\nu,1} \qquad \text{(a.s., q.m.)}.$$

Finally let us examine the case $Z_N M^{-N} \cdot u_{\nu,j}$ $(j = 2, \ldots, j_0(\nu))$. In this case we have also a squared integrable martingale, but we have to note that

$$\begin{aligned} M^{-N} u_{\nu,j} & \\ &= \frac{1}{\rho_\nu^N} y_N \\ &= \frac{1}{\rho_\nu^N} \left(u_{\nu,j} + \sum_{i=1}^{j-1} \frac{(-1)^i [N(N+1)(N+2) \cdots (N+i-1)]}{i!} \right. \\ &\qquad \left. \times \frac{u_{\nu,j-i}}{\rho_\nu^i} \right). \end{aligned}$$

In fact, if we use the expression for $f(J)$ in [5, p. 100] with $f(J) = J^{-N}$, we have then

$$\frac{f^{(i)}(\rho_\nu)}{i!} = \frac{(-1)^i [N(N+1) \cdots (N+i-1)]}{i! \rho_\nu^{N+i}} \qquad (i \geqslant 1),$$

and the result is obvious.

For the associated increasing process we can say

$$A_N = \sum_{n=1}^{N} E\left[\left(Z_n M^{-n} \cdot u_{\nu,j} - Z_{n-1} M^{-(n-1)} \cdot u_{\nu,j}\right)^2 / Z_{n-1}\right]$$

$$= \sum_{n=1}^{N} E\left[\left(\frac{(Z_n - Z_{n-1}M) \cdot y_n}{\rho_n \nu^n}\right)^2 / Z_{n-1}\right]$$

$$= \sum_{n=1}^{N} \frac{Z_{n-1} \cdot b_n}{\rho^{2n}},$$

where b_n is the vector with components $b_n(i) = y_n \Sigma^i y_n^T$ $(i = 1, \dots, r)$, i.e.:

$$b_n(i) = u_{\nu,j} \sum^{i} u_{\nu,j}^T$$

$$+ 2 \sum_{k=1}^{j-1} \frac{(-1)^k [n(n+1) \cdots (n+k-1)]}{k! \rho_\nu^k} u_{\nu,j} \sum^{i} u_{\nu,j-k}^T$$

$$+ \sum_{k=1}^{j-1} \sum_{l=1}^{j-1} \frac{(-1)^{k+l} [n \cdots (n+k-1)][n \cdots (n+l-1)]}{k! l! \rho_\nu^{k+l}}$$

$$\times u_{\nu,j-k} \sum^{i} u_{\nu,j-l}^T.$$

If we look at this expression for $b_n(i)$, we verify that A_N can be written as a sum where the term that might present the most problems with regard to convergence is obviously

$$\frac{1}{[(j-1)!]^2 \rho_\nu^{2j}} \sum_{n=1}^{N} [n(n+1) \cdots (n+j-2)]^2 \frac{Z_{n-1} \cdot \operatorname{var}[Z_1 \cdot u_{\nu,1}]}{\rho_\nu^{2(n-1)}}.$$

As the series $\Sigma n^\alpha a^n$ $(\alpha \geqslant 1, |a| \leqslant 1)$ are convergent, we can reason as in the last case (see Proposition 1), and derive also for $j = 2, \dots, j_0(\nu)$

$$Z_N M^{-N} \cdot u_{\nu,j} \xrightarrow[N \to +\infty]{\text{a.s. q.m.}} W_{\nu,j} < +\infty \qquad \text{(a.s.)}.$$

As we have established that all the components of $Z_N M^{-N}$ on the vectors $v_{\nu,j}$ converge almost surely, we have

$$Z_N M^{-N} \xrightarrow[N \to +\infty]{\text{a.s.}} \sum_{\nu=1}^{\nu_0} \sum_{j=1}^{j_0(\nu)} W_{\nu,j} v_{\nu,j} = T < +\infty \qquad \text{(a.s.)}.$$

Although we will not use the result in the sequel, we note that we have obtained also

$$Z_N M^{-N} \xrightarrow[N \to +\infty]{\text{q.m.}} T \qquad \blacksquare$$

To close this section, we recall some results from linear algebra (see [5] for basic concepts).

Let

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

be a polynomial of degree n ($n \geqslant 1$) with unit coefficient of the principal term. Let $C(f(x))$ be the companion matrix of the polynomial $f(x)$ which is defined by

$$C(f(x)) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix}.$$

It can be easily seen that the characteristic polynomial of $C(f(x))$ is $f(x)$, i.e., the eigenvalues of the companion matrix of $f(x)$ are the roots of $f(x)$.

Let $v \in \mathbb{R}^r$. We define the M-minimal polynomial of v, denoted by P_v, as the monic polynomial $f(\lambda)$ of smallest degree such that

$$f(M)(v) = 0.$$

Remark that if the degree of P_v is k, then

$$v, vM, \ldots, vM^{k-1}$$

are linearly independent. Recall also that the annihilating polynomial of M of least degree with highest coefficient 1 is called the minimum polynomial of M and that a matrix whose minimum polynomial is equal to its characteristic polynomial is called nonderogatory.

PROPOSITION 2.2. *If M is a nonderogatory matrix and*

$$T = \sum_{\nu=1}^{\nu_0} \sum_{j=1}^{j_0(\nu)} a_{\nu,j} v_{\nu,j}, \qquad a_{\nu,1} \neq 0, \quad \nu = 1, \dots, \nu_0,$$

then

$$\begin{pmatrix} T \\ TM \\ \vdots \\ TM^{r-1} \end{pmatrix}$$

is invertible.

Proof. It is easy to see that for every ν

$$v_{\nu,i}(M - \rho_\nu)^{j_0(\nu)} = 0, \qquad i = 1, \dots, j_0(\nu). \tag{4}$$

Moreover

$$v_{\nu,i}(M - \rho_\nu)^{j_0(\nu)-1} \neq 0 \tag{5}$$

if and only if $i = 1$.

Let

$$S_\nu = a_{\nu,1} v_{\nu,1} + a_{\nu,2} v_{\nu,2} + \cdots + a_{\nu,j_0(\nu)} v_{\nu,j_0(\nu)}$$

with $a_{\nu,1} \neq 0$. Using (4), we can see that $S_\nu(M - \rho_\nu)^{j_0(\nu)} = 0$. Using (5), we can see that $S_\nu(M - \rho_\nu)^{j_0(\nu)-1} \neq 0$. Then the minimum polynomial of S_ν is $(\lambda - \rho_\nu)^{j_0(\nu)}$.

Observe that [7, p. 135] if M is a nonderogatory matrix, to each eigenvalue corresponds exactly one Jordan block. Then, since

$$T = S_1 + \cdots + S_{\nu_0},$$

we get, using the Lemma on p. 181 in [5], that

$$P_T = \prod_{\nu=1}^{\nu_0} (\lambda - \rho_\nu)^{j_0(\nu)} = \det(\lambda I_r - M),$$

where I_r denotes the $r \times r$ identity matrix.

Bearing in mind the remark preceding this proposition, we conclude that the vectors

$$T, TM, \dots, TM^{r-1}$$

are linearly independent. ∎

3. THE ESTIMATOR'S ALMOST SURE CONVERGENCE

Let B_N be the matrix whose rows are $Z_N, Z_{N+1}, \dots, Z_{N+r-1}$.

THEOREM 3.1. *If the matrix M is nonderogatory and for all its nonprincipal eigenvalues ζ one has $|\zeta|^2 > \rho$, then the almost sure limit of the sequence $\{B_N M^{-N}\}_{N \geqslant 1}$ exists and is almost surely invertible.*

Proof. First we prove that $B = \lim_{\text{a.s.}} B_N M^{-N}$ exists. Using Lemma 2, we can state, for every $k \in \mathbb{N}$,

$$\lim_{N \to +\infty} Z_{N+k} M^{-N} = \lim_{N \to +\infty} Z_{N+k} M^{(-N+k)} M^k = TM^k \qquad (\text{a.s.}).$$

Then

$$B_N M^{-N} = \begin{pmatrix} Z_N \\ Z_{N+1} \\ \vdots \\ Z_{N+r-1} \end{pmatrix} M^{-N} \xrightarrow[N \to +\infty]{\text{a.s.}} \begin{pmatrix} T \\ TM \\ \vdots \\ TM^{r-1} \end{pmatrix} = B.$$

To see that the matrix B is almost surely invertible we have to notice first that M is a nonderogatory matrix and second that B is a matrix of the type treated on Proposition 2.

To see that $T = \lim_{\text{a.s.}} Z_N M^{-N}$ is a vector satisfying the conditions of this last proposition it is sufficient to note that the random variables $W_{\nu,1}$, components of T on $v_{\nu,1}$ ($\nu = 1, \ldots, \nu_0$), have the following property:

$$P[W_{\nu,1} = 0] = 0 \qquad (\nu = 1, \ldots, \nu_0).$$

In fact, this was the object of the second part of Proposition 1. ■

Consider now the sequence $\{B_N B_{N-1}^{-1}\}_{N \geqslant 1}$, and note that, for each trajectory, B_N will usually be an invertible matrix. (If not, we can skip the corresponding term.)

Assume also, from now on, the hypotheses of the last theorem:

(iii) M is nonderogatory.
(iv) If ζ is a nonprincipal eigenvalue of M, $|\zeta|^2 > \rho$.

THEOREM 3.2. *The sequence* $\{B_N B_{N-1}^{-1}\}_{N \geqslant 1}$ *converges almost surely to the matrix* BMB^{-1}, *which is the companion matrix of the characteristic polynomial of* M.

Proof. We have

$$B_N B_{N-1}^{-1} = B_N M^{-N} M^N B_{N-1}^{-1} \xrightarrow[N \to +\infty]{} BMB^{-1} \qquad (\text{a.s.}),$$

where BMB^{-1} is obviously a matrix similar to M. We have also

$$BMB^{-1} = \begin{pmatrix} TM \\ TM^2 \\ \vdots \\ TM^r \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_r \end{pmatrix},$$

where the $X_1, \ldots, X_r$ are the columns of the inverse of B. Then $TM^{i-1} \cdot X_j = \delta_{ij}$, and we have that

$$BMB^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ TM^r \cdot X_1 & TM^r \cdot X_2 & TM^r \cdot X_3 & \cdots & TM^r \cdot X^r \end{pmatrix}$$

is the companion matrix of the characteristic polynomial of BMB^{-1} and, of course, of M. ■

COROLLARY 3.1. *The sequence* $\{Z_{N+r}B_N^{-1}\}_{N\geq 1}$ *converges almost surely to* $(-a_r, \ldots, -a_1)$, *where* a_i *is coefficient of* $x^{r-i}(i = 1, \ldots, r)$ *in the characteristic polynomial.*

Proof. The conclusion follows immediately from last theorem. In fact,

$$B_{N+1}B_N^{-1} = \begin{pmatrix} Z_{N+1} \\ \vdots \\ Z_{N+r} \end{pmatrix} \begin{pmatrix} Z'_N & \cdots & Z'_{N+r-1} \end{pmatrix},$$

where

$$\begin{pmatrix} Z_N \\ \vdots \\ Z_{N+r-1} \end{pmatrix} \begin{pmatrix} Z'_N & \cdots & Z'_{N+r-1} \end{pmatrix} = I_r.$$

We have then $Z_{N+i} \cdot Z'_{N+j} = \delta_{ij}$ and

$$B_{N+1}B_N^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ Z_{N+r} \cdot Z'_N & Z_{N+r} \cdot Z'_{N+1} & Z_{N+r} \cdot Z'_{N+2} & \cdots & Z_{N+r} \cdot Z'_{N+r-1} \end{pmatrix}.$$

■

Finally we present the estimator and a result on its almost sure convergence.

THEOREM 3.3. *The sequence* $\{$*eigenvalues of* $B_N B_{N-1}^{-1}\}_{N\geq 1}$ *converges almost surely to the eigenvalues of* M.

Proof. As in the case treated on the Introduction, this result is an immediate consequence of the continuity of the function that applies the entries of a square real matrix to its eigenvalues. ■

4. A PRELIMINARY SIMULATION STUDY

To have a first insight into the estimator's behavior we performed some simple simulated examples in the two and three type cases, using Mathematica, version 2.2 for the Macintosh.

We choose reproduction mean matrices with large eigenvalues to allow the convergence to be apparent in the early generations. However, our experience seems to show that, in the case of smaller eigenvalues, essentially the same thing happens in later generations. In fact, it is our belief that, once all the components of the generation vector attain a high level, the estimator approaches the true value even when the spectral condition number of $B_N B_{N-1}^{-1}$ is large, i.e., the problem of determining the eigenvalues of $B_N B_{N-1}^{-1}$ is ill conditioned.

Since branching processes under the just-mentioned conditions become rapidly explosive, a single run can easily take many hours of computing effort even if we restrict ourselves to a small number of generations. Therefore, with a few exceptions where we carried the calculations farther, we had to limit our work to a small number of runs (no more than 15), comprehending each no more than 5 estimates. For that same reason we used small variance reproduction vectors (no correlation between components was considered), which produced runs that did not differ very much between them.

In the following, we present an example of the so far obtained results, emphasizing that a more complete study must be carried out using more and longer runs, specially focusing the effects of larger variances and the role played by the magnitude of the Z_N components in the rate of convergence.

For the two type case, we considered a process with mean reproduction matrix

$$\begin{pmatrix} 7.6 & 2.5 \\ 4.7 & 7.4 \end{pmatrix}$$

and covariance matrices

$$\Sigma^1 = \begin{pmatrix} 0.24 & 0 \\ 0 & 0.25 \end{pmatrix} \quad \text{and} \quad \Sigma^2 = \begin{pmatrix} 0.21 & 0 \\ 0 & 0.24 \end{pmatrix}.$$

The eigenvalues of this mean matrix are 10.9293 and 4.07071.

The results are summarized in the box plots in Figure 1. Though the overall results seem satisfactory, it is apparent from the comparison of the two plots that the convergence in the case of the second eigenvalue is not as good as in the case of the principal eigenvalue. In fact, here, and in all the

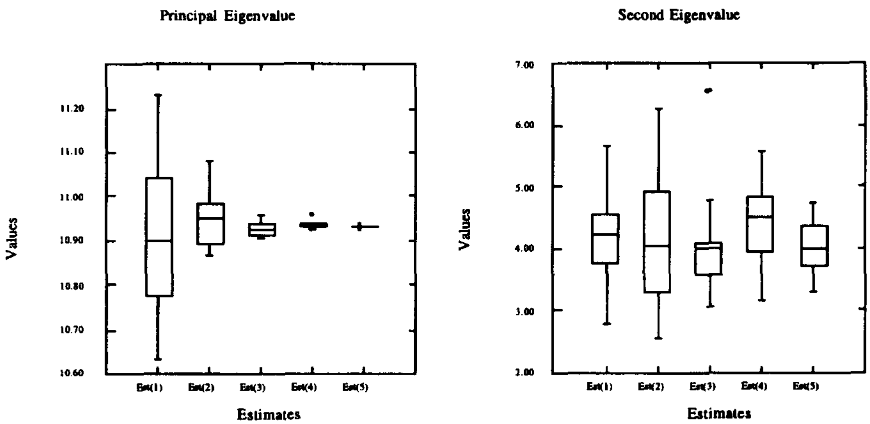

FIG. 1.

examples we performed, the closeness to the true value of the second eigenvalue seemed to be much more affected by the condition number of the matrices B_{N-1} which have to be inverted to evaluate the successive estimates. Therefore, we recommend that, in practice, one should evaluate this condition number along with the estimates and discard the ones for which this number is found to be excessively high.

To test the effect of the condition $|\zeta|^2 > \rho$ of Theorem 1 on the estimator's convergence, we also studied some cases involving matrices with eigenvalues satisfying the opposite inequality. In all the cases the convergence for the principal eigenvalue seemed not affected, while for the second eigenvalue it seemed not to take place.

For the 3×3 case, we obtained essentially the same conclusions, though we performed less runs for the matrices we considered. Here the convergence occurs rapidly for the principal and second eigenvalues, while the estimates of the third experienced some difficulty in stabilizing around the true value, with large deviations when the condition number of B_{N-1} was high.

Finally, we want to point out that, along with the eigenvalues of $B_N B_{N-1}^{-1}$, we evaluated the sequence of matrices $M_N = B_{N-1}^{-1} B_N$. As we were anticipating, the values obtained seemed to approach the matrix M, with remarkable good results in the 2×2 case, especially when the condition number of the matrices to be inverted was low. Therefore, we conjecture that, under the same hypothesis of the main results of the previous section, the just-defined sequence M_N is an almost sure estimator of M, exclusively based upon the knowledge of the successive generation vectors.

If the conjecture proves right, this will be the first example of an estimator for the mean matrix not using the observation of the actual reproductions and could be considered as a multivariate extension of the Lotka estimator for the reproduction mean of the univariate process, $\overline{m} = Z_n/Z_{n-1}$.

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